RELATIVISTIC ELECTRON IN CURVED MAGNETIC FIELDS

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ABSTRACT

Making use of the perturbation method based on the nonlinear differential equation theory, the present work investigates the classical motion of a relativistic electron in a class of curved magnetic fields which may be written as $\vec{B}=\vec{B}(0,B_\varphi,0)$ in cylindrical coordinates (R,φ,Z). Under general astrophysical conditions the author derives the analytical expressions of the motion orbit, pitch angle etc. of the electron in their dependence upon parameters characterizing the magnetic field and electron. The effects of non-zero curvature of magnetic field lines on the motion of electrons and applicabilities of these results to astrophysics are also discussed.

INTRODUCTION

In astrophysics, some curved magnetic fields with sufficiently small field gradients may approximately be written in a local coordinate system as

$$\vec{B}=\vec{B}(0,B_\varphi,0), \qquad B_\varphi = B_0>0, \tag{1}$$

where B_0 is a constant, and (R,φ,Z) denote the cylindrical coordinates. The classical motion of a relativistic electron of charge $-e$ in the field (1) can be exactly solved by the topological method, which was investigated by the author in some detail.[1] The purpose of the present paper is to find further the analytical solution for the motion, and then to extend the results to more general magnetic fields. In the following treatment, the influence of the radiation damping will be neglected.

RESULTS

By virtue of the assumed (without loss of generality) initial position and velocity

$$\vec{r}\Big|_{t=0} = \vec{r}_0(R_0,0,0), \qquad \vec{v}\Big|_{t=0} = \vec{v}_0(v_{R0},v_{\varphi 0},v_{z0}), \tag{2}$$

the first integral of the equation of motion for a relativistic electron in the field (1) may be put into the form

$$d^2R/dt^2+\omega_b^2R = R_0^2v_{\varphi 0}^2/R^3 + \omega_b^2(R_0+v_{z0}/\omega_b), \tag{3}$$

$$R^2d\varphi/dt = R_0v_{\varphi 0}, \tag{4}$$

$$dz/dt = -\omega_b(R-R_0-v_{z0}/\omega_b), \tag{5}$$

in which $\omega_b = eB_0/\gamma mc$ is the relativistic cyclotron frequency, γ the Lorentz factor, and R_0 the curvature radius of the field line passing through the initial point. Using the perturbation method based on Poincare' theory[2] to solve the nonlinear autonomous equations (3)-(5), with the initial conditions (2) and

$$\mu = \beta c/\omega_b R_0 \ll 1, \qquad \beta_{\perp 0} \lesssim |\beta_{/\!/0}|, \tag{6}$$

where

$$\beta = v/c, \qquad \beta_{\perp 0} = (\beta_{R0}^2+\beta_{z0}^2)^{1/2}, \qquad \beta_{/\!/0} = \beta_{\varphi 0},$$

a condition that is adequately met in astrophysics, we get the equations of electron trajectory which, up to and including of the second order in μ, are

$$(R/R_0)-1 \doteq \mu\left[(\beta_{z0}/\beta)+\mu(\beta_{/\!/0}/\beta)^2\right](1-\cos\omega t)+\mu(\beta_{R0}/\beta)\sin\omega t, \tag{7}$$

$$(z-u_c t)/R_0 \doteq \mu\left[(\beta_{z0}/\beta)+\mu(\beta_{/\!/0}/\beta)^2\right]\sin\omega t-\mu(\beta_{R0}/\beta)(1-\cos\omega t), \tag{8}$$

$$\varphi-(1-2\mu\beta_{z0}/\beta)\omega_{/\!/0}t \doteq 2\mu^2(\beta_{/\!/0}/\beta)\left[(\beta_{z0}/\beta)\sin\omega t-(\beta_{R0}/\beta)(1-\cos\omega t)\right], \tag{9}$$

where

$$\omega \doteq (1+3\mu^2\beta_{/\!/0}^2/2\beta^2)\,\omega_b, \qquad u_c \doteq -\mu^2\omega_b R_0\beta_{/\!/0}^2/\beta^2, \qquad \omega_{/\!/0}=v_{\varphi 0}/R_0.$$

On differentiating (3), (4) and (5) with respect to the time, one can obtain further the analytical expression of the electron velocity. It is apparent from (7)-(9) that the motion of the electron in the field (1) may be represented as the superposition of both the helical motion with gyration radius

$$\rho \doteq \mu\left[\frac{\beta_{\perp 0}^2}{\beta^2} + 2\mu\frac{\beta_{z0}}{\beta}\cdot\frac{\beta_{/\!/0}^2-\beta_{\perp 0}^2}{\beta^2} + \mu^2\left(\frac{\beta_{/\!/0}^4}{\beta^4} + \frac{\beta_{z0}^2}{\beta^2}\cdot\frac{\beta_{\perp 0}^2-2\beta_{/\!/0}^2}{\beta^2}\right)\right]^{1/2},$$

and the curvature drift motion with drift velocity u_c.

The pitch angle of an electron moving in a curved magnetic field, ψ, and its mean value $\bar{\psi}$, defined as

$$\bar{\psi} = \sin^{-1}\left[\frac{1}{2\pi}\int_0^{2\pi}\sin^2\psi\, d(\omega t)\right]^{1/2},$$

are customarily calculated in the reference frame where the drift velocity of the electron vanishes. Following this convention, the pitch angle for the electron in the field (1) is found to be

$$\psi \doteq \sin^{-1}\frac{\beta_{\perp 0}}{\beta}\Big[1+\mu\frac{\beta_{z0}\beta_{\parallel 0}^2}{\beta\beta_{\perp 0}^2}(1+\frac{\beta_{z0}^2}{2\beta^2})-\mu\frac{\beta_{\parallel 0}^2}{\beta^2}(\frac{\beta_{z0}}{\beta}\cos\omega t-\frac{\beta_{R0}}{\beta}\sin\omega t)\Big], \tag{10}$$

$$\bar{\psi} \doteq \sin^{-1}(\beta_{\perp 0}/\beta)\big[1+\mu(\beta_{z0}\beta_{\parallel 0}^2/\beta\beta_{\perp 0}^2)(1+\beta_{z0}^2/2\beta^2)\big]. \tag{11}$$

In particular, for an electron in typical curvature motion, corresponding to $\beta_{\perp 0}\ll|\beta_{\parallel 0}|$, the eqs. (10) and (11) reduce to

$$\psi\doteq\bar{\psi}\doteq(\mu^2+2\mu\beta_{z0}/\beta+\beta_{\perp 0}^2/\beta^2)^{1/2}. \tag{12}$$

DISCUSSION

In topics concerned with the properties of the motion and radiation of a relativistic electron in a magnetic field what is taken into account is usually the influence of radiation damping on the motion and pitch angle (the so-called "radiation compression")[3], and sometimes other effects like the magnetic lens. However, our results show that for a relativistic electron in typical curvature motion the influence of the curvature of magnetic field lines is also important.

The latter influence will become quite clear in the special case $v_{\perp 0}=0$ associated with primary particles flying out along magnetic field lines from the surface of pulsars. In this case, from (12), $\psi\doteq\bar{\psi}\doteq\mu$. This indicates that owing to the effect of the non-zero curvature of field lines, the initial motion of the electron, even if the initial transverse velocity vanishes, can not generally be maintained, but have to develop into the helical motion with the pitch angle $\psi\doteq\mu$ and the gyration radius $\rho\doteq\mu^2$, plus the curvature drift. Another special case occurs in $v_{\perp 0}=v_{z0}=-\mu v\beta_{\parallel 0}^2/\beta^2\doteq-\mu v$ for which the pitch angle of the electron is strictly equal to zero.

It can be expected that these results should be conducive to calculating or predicting synchro-curvature radiations from some nonthermal sources, and could exert an influence on the process of the electron momentum distribution "one-dimensionalization" along curved magnetic field lines due to radiation damping. Furthermore, when the effect of radiation damping is taken into account, it may be easily deduced [3] that, as the result of the "radiation compression", a relativistic electron in the field (1) should move approximately along Cornu spiral in the guiding center frame, and tend finally towards the limiting motion corresponding to the latter special case mentioned above.

So far we have found the analytical solutions for a relativistic electron moving in the field (1). Let us consider now the class of curved magnetic fields

$$\vec{B} = \vec{B}(0,\ B_\varphi,\ 0), \qquad B_\varphi/B_0 = (R/R_0)^N \tag{13}$$

(where N is a real constant), which is more general than the field (1) and reduces to (1) at N=0. Most of commen curved magnetic fields with axial symmetry in astrophysics such as the dipole magnetic field may be expressed by (13) in a local frame of refrence if only the magnitude of the field gradient along magnetic field lines is negligibly small. To maintain the fields (13), there must be electric currents flowing in the direction parallel to $\hat{z}$, the density $\vec{j}$ of which is given by

$$\vec{j}/j_0 = (R/R_0)^{N-1}\ \hat{e}_z, \qquad j_0 = (N+1)cB_0/4\pi R_0, \tag{14}$$

where j_0 denotes the current density at the initial point. In application, one can select a configuration of magnetic field from (13), (14) so that it is appropriate for the considered astronomical object. It may be verified that under the condition

$$\begin{aligned} &|N| \ll \mu^{-1}, && \text{for} \quad \beta_{\perp 0} \sim |\beta_{/\!/0}|, \\ &|N| \ll (\mu|\beta_{z0}|/\beta + \mu^2)^{-1}, && \text{for} \quad \beta_{\perp 0} \ll |\beta_{/\!/0}| \end{aligned} \tag{15}$$

preceding results based on the field (1), provided $\omega \doteq \omega_b$, will continue to be valid for the fields (13).

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